A LONG JAMES SPACE

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This paper investigates some of the measurability properties of the James-type Banach space $J(\omega_1)$ obtained with an uncountable ordinal for index set. This space $J(\omega_1)$ is a second dual space with the Radon-Nikodym Property but is not weakly compactly generated. This answers a question of P. Morris reported in [1, p. 87]. (This question has also been answered by W. J. Davis, unpublished.) The space $J(\omega_1)$ is a dual RNP space, but it admits no equivalent weakly locally uniformly convex dual norm. This answers a question in Diestel-Uhl [1, p. 212]. The space $J(\omega_1)$ is a dual RNP space, but there is a bounded, scalarly measurable function on some probability space with values in $J(\omega_1)$ that is not Pettis integrable. The previously known "examples" of this phenomenon depend on the existence of a measurable cardinal [3, Example (1)]. The space is a dual RNP space, but the weak and weak* Borel sets are not the same. This answers a question asked in [10] and [4].

Other properties of this space can be found in the literature. For example, Hagler and Odell [6] have shown that every infinite-dimensional subspace of $J(\omega_1)$ contains an isomorphic copy of ℓ^2.

We will use the following definitions for transfinite series and bases in a Banach space X. Let η be an ordinal, and let $x_\alpha \in X$ be given for each $\alpha < \eta$. The value (when it exists) of the series

$$\sum_{\alpha<\gamma} x_\alpha$$

is defined recursively as follows. If $\gamma = 0$, then

$$\sum_{\alpha<0} x_\alpha = 0 \quad .$$

If $\gamma = \beta+1$ is a successor, then

$$\sum_{\alpha<\gamma} x_\alpha = \sum_{\alpha<\beta} x_\alpha + x_\beta \quad ,$$

provided the series on the right-hand side converges. If γ is a limit, then

$$\sum_{\alpha<\gamma} x_\alpha = \lim_{\beta<\gamma} \Big(\sum_{\alpha<\beta} x_\alpha \Big) \quad ,$$

where the limit is taken in the norm topology of X.

Supported in part by National Science Foundation grant MCS 77-04049 A01.

A transfinite sequence $(x_\alpha)_{\alpha<\eta}$ of vectors is called a basis for X iff for each $y \in X$, there is a unique sequence $(c_\alpha)_{\alpha<\eta}$ of scalars such that

$$y = \sum_{\alpha<\eta} c_\alpha x_\alpha \ .$$

Let η be an ordinal, and let $f\colon [0,\eta] \to \mathbb{R}$ be a function. The square variation of f is

$$(*) \qquad \sup\Big(\sum_{i=1}^{n} |f(\alpha_i) - f(\alpha_{i-1})|^2\Big)^{1/2}$$

where the sup is taken over all finite sequences $\alpha_0 < \alpha_1 < \alpha_2 <\ldots< \alpha_n$ in $[0,\eta]$. Let $J(\eta)$ be the set of all continuous functions f on $[0,\eta]$ with finite square variation and $f(0) = 0$. Then $J(\eta)$ is a Banach space with norm $(*)$. Alternately, let $\tilde{J}(\eta)$ be the set of all functions f on $[0,\eta[$ with finite square variation and $f(0) = 0$. For infinite η, the unique order preserving map of $[0,\eta[$ onto the non-limits in $[0,\eta]$ induces an isometry of $J(\eta)$ onto $\tilde{J}(\eta)$.

We begin by computing a basis for $J(\eta)$. If $\alpha \in [0,\eta]$, define $h_\alpha \in J(\eta)$ by

$$h_\alpha = \chi_{]\alpha,\eta]} \ .$$

Clearly, $\|h_\alpha\| = 1$. Define a projection P_α on $J(\eta)$ by

$$P_\alpha f = f\chi_{[0,\alpha]} + f(\alpha)\chi_{]\alpha,\eta]} \ .$$

PROPOSITION 1. The transfinite sequence $(h_\alpha)_{\alpha<\eta}$ is a basis for the Banach space $J(\eta)$.

Proof: Let $f \in J(\eta)$. I claim first that if γ is a limit ordinal, then $\lim_{\beta<\gamma} \|P_\beta f - P_\gamma f\| = 0$. Let $\epsilon > 0$. There exists a finite sequence $\alpha_0<\alpha_1<\ldots<\alpha_n$ in $[0,\eta]$ with

$$\|P_\gamma f\|^2 < \sum_{i=1}^{n} |P_\gamma f(\alpha_i) - P_\gamma f(\alpha_{i-1})|^2 + \epsilon \ .$$

Since $P_\gamma f$ is constant on $[\gamma,\eta]$, we may assume $\alpha_n \le \gamma$. Since f is continuous at γ, we may assume $\alpha_n < \gamma$. Consider $\beta \in]\alpha_n,\gamma[$. Then the same sequence $\alpha_0 < \alpha_1 <\ldots< \alpha_n$ shows that $\|P_\beta f\|^2 > \|P_\gamma f\|^2 - \epsilon$. Now $P_\beta f$ is constant on $[\beta,\gamma]$ and $(P_\gamma - P_\beta)f$ is constant on $[0,\beta]$, so

$$\|P_\gamma f\|^2 \ge \|P_\beta f\|^2 + \|(P_\gamma - P_\beta)f\|^2 \ .$$

Therefore $\|(P_\gamma - P_\beta)f\|^2 \le \|P_\gamma f\| - \|P_\beta f\| < \epsilon$.

Now given $f \in J(\eta)$, define $c_\alpha = f(\alpha+1) - f(\alpha)$ for $\alpha \in [0,\eta[$. Then

$$f = \sum_{\alpha<\eta} c_\alpha h_\alpha \quad .$$

This is proved by the equation $\sum_{\alpha<\gamma} c_\alpha h_\alpha = P_\gamma f$, which follows by induction on γ. □

COROLLARY. The space $J(\eta)$ is separable if and only if the ordinal η is countable.

Next we consider duals and preduals for $J(\eta)$. For $\alpha \in]0,\eta]$, define $e_\alpha \in J(\eta)^*$ by $e_\alpha(f) = f(\alpha)$. Then $\|e_\alpha\| = 1$.

PROPOSITION 2. The closed linear span Y of $\{e_\alpha : \alpha \in]0,\eta]$, α not a limit ordinal$\}$ is an isometric predual of $J(\eta)$ in the sense that Y^* is isometric to $J(\eta)$.

Proof: The space Y is a norming space of functionals for $J(\eta)$. Indeed, functionals of the form

$$\sum_{i=1}^{n} t_i(e_{\alpha_i} - e_{\alpha_{i-1}})$$

(where $\alpha_0 < \alpha_1 < \ldots < \alpha_n$ are non-limits and $\Sigma|t_i|^2 \le 1$) have norm 1 and norm $J(\eta)$ isometrically. The unit ball $B = \{f \in J(\eta) : \|f\| \le 1\}$ is compact in the topology of pointwise convergence on $\{e_\alpha : \alpha$ not a limit$\}$. To see this, consider a net f_θ in B . By taking a subnet, we may assume $f_\theta(\alpha)$ converges for all non-limits α , call that limit $f(\alpha)$. If $\alpha_0 < \alpha_1 < \ldots < \alpha_n$ are non-limits, then

$$\sum_{i=1}^{n} |f(\alpha_i) - f(\alpha_{i-1})|^2 \le 1 \quad .$$

From this it follows that the limits $\lim_{\alpha<\beta} f(\alpha)$ exist for limit ordinals β , call these limits $f(\beta)$. Then $f \in B$ and $f_\theta \to f$ pointwise on the non-limits.

Finally, since B is bounded, it is compact in the topology $\sigma(J(\eta),Y)$. Therefore $J(\eta) = Y^*$ isometrically (cf. [2, V.5.7]). □

We will see below that $J(\eta)^*$ has the Radon-Nikodym property. It follows from this that the isometric predual is unique [5].

PROPOSITION 3. The sequence $(e_\alpha)_{\alpha\in]0,\eta]}$ is a basis for $J(\eta)^*$.

Proof: Let $\ell \in J(\eta)^*$. We first claim that if γ is a limit, then $\lim_{\beta<\gamma}\ell(h_\beta)$ exists. Suppose it does not exist. Then there are real numbers $a < b$ and ordinals $\beta_0 < \beta_1 < \beta_2 < \ldots < \gamma$ with $\ell(h_{\beta_{2i}}) < a$, $\ell(h_{\beta_{2i+1}}) > b$. But for each n, we have $\|\sum_{i=1}^{n} (h_{\beta_{2i-1}} - h_{\beta_{2i}})\| = (2n)^{1/2}$, so

$$n(b-a) < \ell(\sum_{i=1}^{n} (h_{\beta_{2i-1}} - h_{\beta_{2i}}))$$
$$\leq \|\ell\|(2n)^{1/2} ,$$

so $\|\ell\| = \infty$, a contradiction.

Define, for $\gamma \in]0,\eta]$,

$$u_\gamma = \begin{cases} \ell(h_{\gamma-1}) , & \gamma \text{ non-limit} \\ \lim_{\beta<\gamma} \ell(h_\beta) , & \gamma \text{ limit} . \end{cases}$$

Then $\lim_{\beta<\gamma} u_\beta = u_\gamma$ for limit ordinals γ. I claim that $\ell = \sum_{\alpha\in]0,\eta]} (u_\alpha - u_{\alpha+1})e_\alpha$. The series converges weak* to ℓ, so it is only required to show that the partial sums converge in norm at any limit ordinal γ. This calculation is the same as the one which shows the basis for the original James space is shrinking. See, for example, [9, p. 274, (d)]. □

Propositions 2 and 3 can be used to describe the canonical embedding of $J(\eta)$ into $J(\eta)^{**}$. In fact (if η is infinite), $J(\eta)^{**}$ is isometric to $\tilde{J}(\eta+1)$, and the set-theoretic inclusion $J(\eta) \to \tilde{J}(\eta+1)$ is the canonical embedding. This shows that $J(\eta)^{**}$ is isometric to $J(\eta+1)$, and isomorphic to $J(\eta)$ itself.

COROLLARY. $J(\eta)^*$ is separable if and only if η is countable.

With the understanding of $J(\eta)$ provided above, many of its properties can be determined.

PROPOSITION 4. The space $J(\eta)$ has the Radon-Nikodym property.

Proof: Consider the predual Y given in Proposition 2. By a result of Uhl [1, p. 82, Cor. 6], it suffices to show that every separable subspace of Y has separable dual. Let Z be a separable subspace of Y. Each element of Z is in the closed span of a countable set of e_α, so there is a countable set $R \subseteq [0,\eta]$ of nonlimits such that $Y_1 = \text{cl sp}\{e_\alpha : \alpha \in R\}$ contains Z. Then the closure $\overline{R}$ is also countable; let η_1 be its order type. Then Y_1^* is isometric to $J(\eta_1)$, which is separable. □

PROPOSITION 5. The dual $J(\eta)^*$ has the Radon-Nikodym property.

PROOF: Any separable subspace of $J(\eta)$ is in the closed span of countably many vectors h_α. Therefore, as above, the dual of such a closed span is isometric to $J(\eta_1)^*$ for some countable ordinal η_1. □

PROPOSITION 6. If $\ell \in J(\eta)^*$, then ℓ is a Borel function on $(J(\eta), \text{weak*})$.

Proof: By Proposition 3, it suffices to show that e_γ is weak*-Borel for all $\gamma \in]0,\eta]$. If γ is a non-limit, then e_γ is weak*-continuous. Assume γ is a limit. The restriction map of $J(\eta)$ onto $J(\gamma)$ is weak*-continuous, so we may assume $\gamma = \eta$. For $\lambda \in \mathbb{R}$, $k \in \mathbb{N}$, $r \in \mathbb{Q}$, define

$$P_1(r) = \{f \in J(\eta) : \|f\| \le r\}$$

$$P_2(r,k,\lambda) = \cup\{f \in J(\eta) : \sum_{i=1}^{n} |f(\alpha_i) - f(\alpha_{i-1})|^2 > r^2 - \frac{1}{k^2},\ f(\alpha_n) < \lambda - \frac{1}{k}\},$$

where the union is over all finite sequences $\alpha_0 < \alpha_1 < \ldots < \alpha_n$ of non-limits, and

$$P(r,k,\lambda) = P_1(r) \cap P_2(r,k,\lambda) \quad .$$

Then $P_1(r)$ is weak*-closed and $P_2(r,k,\lambda)$ is weak*-open. But

$$\{f \in J(\eta) : f(\eta) < \lambda\} = \bigcap_{k=1}^{\infty} \bigcup_{r>0} P(r,k,\lambda),$$

so $\{f : f(\eta) < \lambda\}$ is weak*-Borel. □

PROPOSITION 7. Suppose $\eta \ge \omega_1$, the least uncountable ordinal. Then e_{ω_1} is not a weak*-Baire function.

Proof: The set $R = \{h_\alpha : \alpha \in [0,\omega_1]\}$ is weak*-homeomorphic to $[0,\omega_1]$. Any real-valued continuous function on $[0,\omega_1]$ is constant on some interval $[\gamma,\omega_1]$ with $\gamma < \omega_1$, so any Baire function shares this property. But $e_{\omega_1}(h_\alpha) = 1$ for $\alpha < \omega_1$ and $e_{\omega_1}(h_{\omega_1}) = 0$. Therefore, e_{ω_1} is not a Baire function on R, and a fortiori on $J(\eta)$. □

Since $J(\eta)$ has the Radon-Nikodym property, the weak and weak*-universally measurable sets coincide [8], [3, Theorem 1.5]. For this reason, the following is somewhat surprising.

PROPOSITION 8. There is a weak-Borel subset of $J(\omega_1)$ which is not weak*-Borel.

Proof: Let $R = \{h_\alpha : \alpha \in [0,\omega_1]\}$. Then $(R,\text{weak*})$ is homeomorphic to $[0,\omega_1]$. But

$$\{f \in R: (e_{\alpha+1}-e_\alpha)(f) > \tfrac{1}{2}\} = \{h_\alpha\} , \alpha < \omega_1$$

$$\{f \in R: e_{\omega_1}(f) < \tfrac{1}{2}\} = \{h_{\omega_1}\} ,$$

so (R,weak) is discrete. So every subset of R is a weak-open set. But there is a subset of R which is not weak*-Borel. (A subset $A \subseteq [0,\omega_1[$ such that neither A nor its complement contains a closed unbounded set is not Borel.) □

Although the statement of the following proposition does not involve measurability questions, they are helpful in the proof.

PROPOSITION 9. The space $J(\omega_1)$ admits no equivalent dual norm that is weakly locally uniformly convex.

Proof: In a dual space with weakly locally uniformly convex norm, the weak and weak* topologies coincide on the surface of the unit ball. But then by [4, Theorem 2.1] the weak and weak* Borel algebras coincide on the entire space. So by Proposition 8, the space $J(\omega_1)$ admits no such norm. □

The terms used in the following can be found in [4].

PROPOSITION 10. The space $J(\omega_1)$ is not realcompact, not measure-compact, not Lindelof, not weakly compactly generated, not isomorphic to a subspace of a weakly compactly generated space, and fails the Pettis integral property.

Proof: We show that $J(\omega_1)$ is not realcompact; the other assertions follow from this. Define a zero-one measure μ on Baire $([0,\omega_1[)$ by $\mu(B) = 0$ iff B is countable, $\mu(B) = 1$ iff $[0,\omega_1[\setminus B$ is countable. The map $\varphi: [0,\omega_1[\to J(\omega_1)$ defined by $\varphi(\alpha) = h_\alpha$ is scalarly measurable since $e_\beta \circ \varphi$ is constant a.e. for each β . The image $\lambda = \varphi(\mu)$ is a zero-one measure on Baire $(J(\omega_1),\text{weak})$. But λ is not τ-smooth: If $A \subset [0,\omega_1[$ is countable, then

$$Z_A = \{f \in J(\omega_1): f(\alpha) = 0 \text{ for all } \alpha \in A,\ f(\omega_1) = 1\}$$

is a weak-zero-set, and $\lambda(Z_A) = 1$. But the collection Z_A decreases to $\emptyset$. Thus $(J(\omega_1),\text{weak})$ is not realcompact. □

If the continuum hypothesis holds, then there is a bijection $\theta: [0,1] \to [0,\omega_1[$, and $\varphi \circ \theta$ provides an example of a bounded, scalarly measurable function on $[0,1]$ which is not Pettis integrable with respect to Lebesgue measure, that is, $J(\omega_1)$ fails the Lebesgue-PIP. Similarly, if Martin's Axiom holds, $J(\omega_c)$ fails the Lebesgue-PIP, where ω_c is the least ordinal of power c .

Using the same measure space $[0,\omega_1[$ and the map $\alpha \to e_{\alpha+1}$, it can be shown similarly that the predual Y of $J(\omega_1)$ is not realcompact.

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